

Equivariant torsion and G -CW-complexes

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1 Introduction

1.1

In this note we consider equivariant Reidemeister and analytic torsion invariants of closed oriented G -manifolds, where G is any compact Lie group.

Equivariant analytic torsion for closed oriented odd-dimensional G -manifolds for arbitrary compact Lie groups G and equivariant Reidemeister torsion of closed oriented G -manifolds for finite G were introduced in [7] and further studied in [10] and [1]. In the present paper we generalize the definition of equivariant Reidemeister torsion to general compact Lie groups G and address the question of equality with equivariant analytic torsion.

If G is finite, then equivariant Reidemeister torsion is in fact an invariant of G -equivariant locally constant sheaves \mathcal{F} of finite-dimensional Hilbert spaces over G -CW-complexes. We extend this invariant to general compact Lie groups G .

If M is a closed G -manifold (G a compact Lie group), then there is a natural equivalence class of G -homotopy equivalences $f : X \rightarrow M$ called simple structure (see section 2.2 for details), where X is a G -CW-complex. If \mathcal{F} is a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces, then the equivariant Reidemeister torsion of (M, \mathcal{F}) is defined using $(X, f^*\mathcal{F})$, and it is independent of choices.

Viewing a G -CW complex X as a filtered G -space, we express the equivariant Reidemeister torsion of (X, \mathcal{F}) in terms of the equivariant Reidemeister torsion of the restriction of \mathcal{F} to the G -cells and a contribution of the spectral sequence induced by the filtration. There is a clear separation into an invariant which only depends on restrictions of \mathcal{F} to the G -cells, and an invariant which depends on the way the cells are glued together. If G is connected, then the latter invariant is trivial.

We compute the equivariant Reidemeister torsion and the equivariant analytic torsion in terms of contributions of G -cells. The contribution of a G -cell can be further evaluated by restricting to one-dimensional subgroups of G . In particular we compute the equivariant Reidemeister torsion and the equivariant analytic torsion of compact symmetric spaces by topological means and recover the result of [6]

1.2

Let G be a compact Lie group, and M be a closed odd-dimensional oriented G -manifold. Let $F \rightarrow M$ be a G -equivariant flat hermitean vector bundle and \mathcal{F} be the associated G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces.

If we choose a G -equivariant Riemannian metric g^M , then one defines the equivariant analytic torsion $\rho_{an}(M, g^M, \mathcal{F}) : G \rightarrow \mathbf{C}$ (see [7], §X) as a spectral invariant of the Laplace operator Δ_{g^M} acting on F -valued forms:

$$\rho_{an}(M, g^M, \mathcal{F})(g) := \frac{d}{ds} \Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}_s N g e^{-t\Delta_{g^M}} - \chi'(M, \mathcal{F})(g)) t^{s-1} dt ,$$

where $\chi'(M, \mathcal{F}) := \sum_{i=0}^\infty (-1)^i i \mathrm{Tr} g|_{H^i(M, \mathcal{F})}$, N denotes the \mathbf{Z} -grading of the bundle of F -valued forms, and the integral converges for $\mathrm{Re}(s) \gg 0$ and has a meromorphic continuation to all of \mathbf{C} .

By definition $\rho_{an}(M, g^M, \mathcal{F})$ is a class function on G . If \mathcal{F} is acyclic, i.e. $H^*(M, \mathcal{F}) = 0$, then $\rho_{an}(M, g^M, \mathcal{F})$ is independent of g^M , and we write $\rho_{an}(M, g^M, \mathcal{F}) =: \rho_{an}(M, \mathcal{F})$.

1.3

Let G be finite, let M be a closed oriented G -manifold, and let \mathcal{F} be a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We choose a G -Hilbert module structure on $H^*(M, \mathcal{F})$. Using a smooth G -equivariant triangulation of M one can define

equivariant Reidemeister torsion $\rho(M, \mathcal{F}) : G \rightarrow \mathbf{C}$ which is again a class function on G (see [7], Sec. 5). Equivalently, there is a natural simple structure $f : X \rightarrow M$ and we can define $\rho(M, \mathcal{F}) := \rho(X, f^* \mathcal{F})$. We present the details of this definition in Section 3.

Assume that M is odd-dimensional. If \mathcal{F} is acyclic, then by [7], Prop.16, we have the equality of class functions

$$\rho(M, \mathcal{F}) = \rho_{an}(M, \mathcal{F}) . \quad (1)$$

In case that \mathcal{F} is not acyclic we fix a G -invariant Riemannian metric g^M . Let $\mathcal{H}(M, g^M, \mathcal{F})$ denote the space of harmonic F -valued forms. Then $\mathcal{H}(M, g^M, \mathcal{F})$ is a G -Hilbert module. On $H^*(M, \mathcal{F})$ we choose the G -Hilbert module structure such that the de Rham isomorphism $\mathcal{H}(M, g^M, \mathcal{F}) \xrightarrow{\sim} H^*(M, \mathcal{F})$ becomes an isometry. Again by [7], Prop.16, we have the equality

$$\rho(M, \mathcal{F}) = \rho_{an}(M, g^M, \mathcal{F}) \quad (2)$$

of class functions.

1.4

Let G be any compact Lie group, M be a closed oriented G -manifold, and $F \rightarrow M$ be a G -equivariant flat hermitean vector bundle. We fix a G -Hilbert module structure on $H^*(M, \mathcal{F})$.

In the present subsection we define equivariant Reidemeister torsion of (M, \mathcal{F}) and discuss the validity of (1) and (2).

Let $FG := \{g \in G \mid (\exists n \in \mathbf{N} \mid g^n = 1)\}$ denote the set of elements of finite order. FG is a dense subset of G which is invariant under conjugation. Let $C(FG) := \{f : FG \rightarrow \mathbf{C} \mid f(g^h) = f(g), \forall g \in FG, h \in G\}$ denote the space of all real-valued functions on FG , which are invariant under conjugation, where $g^h := hgh^{-1}$.

Let $\text{Or}_f(G)$ denote the full subcategory of the orbit category (see [8], 8.16) consisting of all homogeneous spaces G/Γ , where $\Gamma \subset G$ is finite.

We have a contravariant functor $C : \text{Or}_f(G) \rightarrow \mathbf{C} - \text{vect}$ associating to the object G/Γ the space of class functions $C(\Gamma)$ on Γ . If $f : G/\Gamma \rightarrow G/\Gamma'$ is a morphism in $\text{Or}_f(G)$, then there is a $g \in G$ such that $\{g\gamma g^{-1} \mid \gamma \in \Gamma\} = \Gamma^g \subset \Gamma'$ and $f(h\Gamma) = hg^{-1}\Gamma'$. The functor C associates to f the map $C(f) : C(\Gamma') \rightarrow C(\Gamma^g) \rightarrow C(\Gamma)$, where the first arrow $\text{res}_{\Gamma^g}^{\Gamma'} : C(\Gamma') \rightarrow C(\Gamma^g)$ is the restriction of class functions and the second is induced by the map $\Gamma \rightarrow \Gamma^g, \gamma \mapsto g\gamma g^{-1}$.

There is a natural bijection

$$C(FG) \xrightarrow{\sim} \lim_{\substack{\text{Or}_f(G) \\ \leftarrow}} C(\Gamma) , \quad (3)$$

which is induced by the restrictions $\text{res}_{\Gamma}^G : C(FG) \rightarrow C(\Gamma)$, $\Gamma \subset G$ finite.

For $\Gamma \subset G$ let $\text{res}_{\Gamma}^G M$ denote the Γ -manifold obtained from M by restricting the G -action to Γ . If V is a G -module, then let $\text{res}_{\Gamma}^G V$ denote the Γ -module obtained by restriction. Note that $H^*(\text{res}_{\Gamma}^G M, \mathcal{F}) = \text{res}_{\Gamma}^G H^*(M, \mathcal{F})$ canonically, and the latter has a natural Γ -Hilbert module structure.

If $G/\Gamma \in \text{Or}_f(G)$, then $\rho(\text{res}_{\Gamma}^G M, \mathcal{F}) \in C(\Gamma)$ is well defined by 1.3. By Proposition 3.5 the collection $\{\rho(\text{res}_{\Gamma}^G M, \mathcal{F})\}_{G/\Gamma \in \text{Or}_f(G)}$ is a section of the functor C and thus defines an element

$$\tilde{\rho}(M, \mathcal{F}) \in \lim_{\substack{\text{Or}_f(G) \\ \leftarrow}} C(\Gamma) .$$

Definition 1.1 *Let $\rho(M, \mathcal{F}) \in C(FG)$ be the element which corresponds to $\tilde{\rho}(M, \mathcal{F})$ under the bijection (3).*

With this definition equality of equivariant Reidemeister and analytic torsion for arbitrary Lie groups becomes a formal consequence of the corresponding result for finite groups.

Lemma 1.2 *Let G be a compact Lie group, M be a closed odd-dimensional oriented G manifold, g^M be a G -invariant Riemannian metric and $F \rightarrow M$ be a G -equivariant flat hermitean vector bundle. We equip $H^*(M, \mathcal{F})$ with the G -Hilbert module structure such that the de Rham isomorphism $\mathcal{H}^*(M, g^M, \mathcal{F}) \cong H^*(M, \mathcal{F})$ becomes an isometry. Then*

$$\rho(M, \mathcal{F}) = \rho_{an}(M, g^M, \mathcal{F})|_{FG} .$$

Proof. Let $g \in FG$ and $G/\Gamma \in \text{Or}_f(G)$ be such that $g \in \Gamma$. Then obviously $\rho_{an}(M, g^M, \mathcal{F})(g) = \rho_{an}(\text{res}_\Gamma^G M, g^M, \mathcal{F})(g)$. By (2) we have

$$\rho(M, \mathcal{F})(g) \stackrel{\text{def}}{=} \rho(\text{res}_\Gamma^G M, \mathcal{F})(g) = \rho_{an}(\text{res}_\Gamma^G M, g^M, \mathcal{F})(g) = \rho_{an}(M, g^M, \mathcal{F})(g) .$$

□

1.5

Note that equivariant Reidemeister torsion depends on the choice of a G -Hilbert module structure on the cohomology $H^*(M, \mathcal{F})$. In the present subsection we show how one can define an invariant that is independent of this choice.

Let $G^0 \subset G$ denote the component of the identity of G . Then we have an exact sequence

$$0 \rightarrow G^0 \rightarrow G \xrightarrow{q} \pi_0(G) \rightarrow 0 .$$

Note that the restriction $q|_{FG} \rightarrow \pi_0(G)$ is still surjective. Thus the pull-back in the sequence below defining $\hat{C}(FG)$ is injective.

$$0 \rightarrow C(\pi_0(G)) \xrightarrow{q^*} C(FG) \rightarrow \hat{C}(FG) \rightarrow 0$$

For any $G/\Gamma \in \text{Or}_f(G)$ let $\Gamma^0 := \Gamma \cap G^0$. Then we have exact sequences

$$\begin{aligned} 0 \rightarrow \Gamma^0 \rightarrow \Gamma \xrightarrow{q} \Gamma/\Gamma^0 \rightarrow 0 \\ 0 \rightarrow C(\Gamma/\Gamma^0) \xrightarrow{q^*} C(\Gamma) \rightarrow \hat{C}(\Gamma) \rightarrow 0 , \end{aligned}$$

where $\hat{C}(\Gamma)$ is defined by the second sequence.

We can consider the functor $\hat{C} : \text{Or}_f(G) \rightarrow \mathbf{C} - \text{vect}$, which associates to $G/\Gamma \in \text{Or}_f(G)$ the space $\hat{C}(\Gamma)$. If $f : G/\Gamma \rightarrow G/\Gamma'$ is a morphism in $\text{Or}_f(G)$, then the map $\hat{C}(f) : \hat{C}(G/\Gamma') \rightarrow \hat{C}(G/\Gamma)$ is represented by $C(f) : C(\Gamma) \rightarrow C(\Gamma')$ which maps $C(\Gamma/\Gamma^0)$ to $C(\Gamma'/(\Gamma')^0)$.

Since the natural map $C(\pi_0(G)) \rightarrow \lim_{\leftarrow \text{Or}_f(G)} C(\Gamma/\Gamma^0)$ is an isomorphism we conclude from

$$\begin{array}{ccccccc} 0 & \rightarrow & C(\pi_0(G)) & \rightarrow & C(FG) & \rightarrow & \hat{C}(FG) \rightarrow 0 \\ & & \downarrow \parallel & & \downarrow \parallel & & \downarrow \hat{I} \\ 0 & \rightarrow & \lim_{\leftarrow \text{Or}_f(G)} C(\Gamma/\Gamma^0) & \rightarrow & \lim_{\leftarrow \text{Or}_f(G)} C(\Gamma) & \rightarrow & \lim_{\leftarrow \text{Or}_f(G)} \hat{C}(\Gamma) \rightarrow \lim_{\leftarrow \text{Or}_f(G)}^1 C(\Gamma/\Gamma^0) \end{array}$$

that \hat{I} is injective.

Note that G^0 acts trivially on $H^*(M, \mathcal{F})$. Thus if $G/\Gamma \in \text{Or}_f(G)$, then Γ^0 acts trivially on $H^*(\text{res}_\Gamma^G M, \mathcal{F})$, too. By Lemma 3.1 the class $\hat{\rho}(\text{res}_\Gamma^G M, \mathcal{F}) \in \hat{C}(\Gamma)$ of $\rho(\text{res}_\Gamma^G M, \mathcal{F}) \in C(\Gamma)$ is independent of the choice of a Γ -Hilbert module structure on $H^*(\text{res}_\Gamma^G M, \mathcal{F})$.

The collection $\{\hat{\rho}(\text{res}_\Gamma^G M, \mathcal{F})\}_{G/\Gamma \in \text{Or}_f(G)}$ defines a section of the functor \hat{C} and therefore an element

$$\tilde{\rho}(M, \mathcal{F}) \in \lim_{\leftarrow \text{Or}_f(G)} \hat{C}(\Gamma) .$$

It is easy to see that $\tilde{\rho}(M, \mathcal{F})$ is in the range of \hat{I} .

Definition 1.3 *Let $\hat{\rho}(M, \mathcal{F}) \in \hat{C}(FG)$ be the unique element such that $\hat{I}(\hat{\rho}(M, \mathcal{F})) = \tilde{\rho}(M, \mathcal{F})$.*

The class $\hat{\rho}_{an}(M, g^M, \mathcal{F}) \in \hat{C}(FG)$ of $\rho_{an}(M, g^M, \mathcal{F})|_{FG}$ is independent of the choice of the G -invariant Riemannian metric g^M (see [7], §X). We thus write $\hat{\rho}_{an}(M, \mathcal{F}) := \hat{\rho}_{an}(M, g^M, \mathcal{F})$. The proof of the following Lemma is similar to that of Lemma 1.2.

Lemma 1.4 *Let G be a compact Lie group, M be a closed odd-dimensional G manifold, and $F \rightarrow M$ be a G -equivariant flat hermitean vector bundle. Then*

$$\hat{\rho}(M, \mathcal{F}) = \hat{\rho}_{an}(M, \mathcal{F}) . \quad (4)$$

1.6

It is natural to ask what kind of differential-topological information about the G -manifold M and the flat bundle F is encoded in the invariants $\rho(M, \mathcal{F})$ and $\hat{\rho}(M, \mathcal{F})$. While $\rho(M, \mathcal{F})$ contains global information about M it turns out that $\hat{\rho}(M, \mathcal{F})$ only depends on the type of G -cells of X and the restriction of $f^*\mathcal{F}$ to the cells, where $f : X \rightarrow M$ represents the preferred simple structure of the smooth closed G -manifold M . In particular it is independent of the way the cells are glued together.

We now formulate the result in detail. A G -space $G/H \times D^n$ is called a n -dimensional G -cell of type H , where $H \subset G$ is a closed subgroup. Let X be a finite G -CW-complex and \mathcal{F} be a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces over X . For any G -cell $E = G/H_E \times D^n \hookrightarrow X$ of dimension $\dim(E) := n$ let $\mathcal{F}_E \rightarrow G/H_E$ denote the restriction of \mathcal{F} to $G/H_E \times \{0\}$. Then $\hat{\rho}(G/H_E, \mathcal{F}_E) \in \hat{C}(FG)$ is defined.

Proposition 1.5 (Corollary 4.4) *Let X be a finite G -CW-complex, and \mathcal{F} be a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces over X . Then we have*

$$\hat{\rho}(X, \mathcal{F}) = \sum (-1)^{\dim(E)} \hat{\rho}(G/H_E, \mathcal{F}_E) ,$$

where the sum is taken over all G -cells of X .

1.7

The equivariant torsion $\hat{\rho}(M, \mathcal{F})$ is determined by its restrictions to all Cartan subgroups T of G . If we apply Proposition 1.5 to $\text{res}_T^G M$, then we can compute the equivariant

torsion of M in terms of the T -cells. In Section 5.2 we study the equivariant torsion of a T -cell. It vanishes iff the isotropy group has codimension zero or greater than one, and it is explicitly computable, if the isotropy group has codimension one.

Let $f : X \rightarrow \text{res}_T^G M$ represent the preferred simple structure. Consider $t \in T$. Let I be the collection of T -cells $E \cong T/S_E \times D^{n_E}$ of X with $\dim(T/S_E) = 1$. For $E \in I$ let $J_E(t)$ be the collection of connected components E_i , $i \in J_E(t)$, of E (note that $E_i \cong S^1 \times D^{n_E}$), such that $tE_i = E_i$. Let $m = \dim(\mathcal{F})$ and $U_i \in U(m)$ be the holonomy of $f^*\mathcal{F}|_{E_i}$. Then U_i is determined uniquely by the choice of a base point $o_i \in E_i$, identification of the fibre \mathcal{F}_{o_i} with \mathbf{C}^m , and the choice of an orientation of E_i . The matrix U_i can be written as $e^{2\pi i a_i}$ for a selfadjoint $a_i \in \text{Mat}(m, \mathbf{C})$. The element t acts as rotation of the circle-part of E_i by the angle $2\pi\tau_i$. Moreover there are unitary isomorphisms λ_i of \mathcal{F}_{o_i} given by the action of t composed with parallel transport back to o_i in direction opposite to the orientation. Note that λ_i and a_i commute. Then

Lemma 1.6 *The equivariant torsion $\hat{\rho}(\text{res}_T^G M, \mathcal{F})$ is given by the class of*

$$T \ni t \mapsto \sum_{E \in I, i \in I_E} (-1)^{n_E} \text{Tr } \psi(\lambda_i, a_i, \tau_i) ,$$

where an explicit formula for ψ is given in 5.4.

For the purpose of illustration in 5.6 we compute the equivariant Reidemeister torsion of odd-dimensional symmetric spaces. Employing (4) we essentially recover the results of [6] about equivariant analytic torsion of symmetric spaces.

2 Restriction of simple structures

In this section we recall some basic results in equivariant topology.

2.1

The G -space $G/H \times D^n$ is called a n -dimensional G -cell of type H , where $H \subset G$ is a closed subgroup. A finite relative G -CW-complex is a pair of G -spaces (X, A) together with a finite filtration by G -spaces

$$A = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_N = X, \quad \cup_{n=-1}^N X_n = X,$$

and a collection of G -subspaces $e_i^n \subset X_n$, $i \in I_n$, $n \geq 0$, $\#I_n < \infty$, with the following properties

- (1) : A/G is Hausdorff
- (2) : X has the weak topology with respect to the filtration $\{X_n\}$
- (3) : for $n \geq 0$ there are G -push outs

$$\begin{array}{ccc} \sqcup_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\sqcup_i q_i} & X_{n-1} \\ \downarrow & & \downarrow \\ \sqcup_{i \in I_n} G/H_i \times D^n & \xrightarrow{\sqcup_i Q_i} & X_n \end{array},$$

such that $e_i^n = Q_i(G/H_i \times \text{int } D^n)$ (see [8] Def. 1.2.).

2.2

If X is a G -space, then a simple structure on X is given by a pair (Z, f) , where Z is a finite G -CW-complex and $f : Z \rightarrow X$ is a G -homotopy equivalence. A second pair (Z', f') defines the same simple structure on X , if $f'_* \tau^G((f')^{-1} \circ f) = 0$ holds true in the Whitehead group $Wh^G(X)$, where $(f')^{-1}$ denotes any homotopy inverse of f' (see [8], 4.27).

Let

$$\begin{array}{ccc} X_0 & \rightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \rightarrow & X \end{array}$$

be a push-out of G -spaces and each of the X_i , $i = 0, 1, 2$ be equipped with a simple structure. Then X has a preferred simple structure [8], p75.

2.3

If X is a closed smooth G -manifold (possibly with boundary), then it has a preferred simple structure ([8], 4.36). It is obtained by induction over the number of orbit types. If X has one orbit type, then X/G is a smooth manifold which has a smooth triangulation. Lifting this triangulation to X we obtain a G -CW-decomposition of X representing the preferred simple structure. If X has several orbit types then we write X as a push-out of two G -manifolds with less orbit types. We apply the induction hypothesis and 2.2 in order to obtain the preferred simple structure of X .

2.4

Let G be a compact Lie group and X be a finite G -CW complex. If $H \subset G$ is a closed subgroup, then in general there is no natural H -CW structure on $\text{res}_H^G X$. But we have the following result.

Proposition 2.1 (1) : *Let X be a finite G -CW complex. If $H \subset G$ is a closed subgroup, then there exists a preferred simple structure $f : Z \rightarrow \text{res}_H^G X$.*

(2) : *Let X be a finite G -CW complex. If $K \subset H \subset G$ are closed subgroups, $f : Z \rightarrow \text{res}_H^G X$ and $g : Y \rightarrow \text{res}_K^H Z$ represent the preferred simple structures given in (1), then $f \circ g : Y \rightarrow \text{res}_K^G X$ represents the preferred simple structure, too.*

(3) : *Let M be a smooth closed G -manifold, and let $f : X \rightarrow M$ represent the preferred simple structure. Let $H \subset G$ be a closed subgroup, and let $g : Z \rightarrow \text{res}_H^G X$ represent the preferred simple structure. Then $f \circ g : Z \rightarrow \text{res}_H^G M$ represents the preferred simple structure of the closed smooth H -manifold $\text{res}_H^G M$.*

(4) : *Let X be a finite G -CW complex, $H \subset G$ be a closed subgroup, and let $f : Z \rightarrow \text{res}_H^G X$ represent the preferred simple structure. Let $g \in G$, $H^g := gHg^{-1}$, Z^g be the H^g -CW complex which is obtained from Z by letting $u \in H^g$ act by $g^{-1}ug$, and define $f^g : Z^g \rightarrow \text{res}_{H^g}^G X$ by $f^g = g \circ f$. Then $f^g : Z^g \rightarrow \text{res}_{H^g}^G X$ represents the preferred simple structure.*

Proof. (1) The homogeneous spaces G/H for all closed subgroups $L \subset G$ are smooth H -manifolds. Thus we have preferred simple structures on the G -cells of X considered as H -spaces. Writing X as a push-out over its G -cells and using 2.2 we obtain a preferred simple structure on $\text{res}_H^G X$. Note that the technical assumptions [8], 7.3 and 7.23 for this procedure are satisfied (see [8], 7.27, see also [3],[4],[5]).

(2) It suffices to show this for the homogeneous spaces G/L . In this case we can apply

(3) (Transitivity of the restriction was also announced in [5]).

(3) This is [8], Lemma 7.4.5.

(4) It again suffices to verify this assertion for the homogeneous spaces G/L . In this case we can apply [5], Lemma 1.4. \square

3 Equivariant torsion

3.1

Let Γ be a finite group, and let $R(\Gamma)$ denote the representation ring of Γ with real coefficients. If π is a finite-dimensional representation of Γ , then χ_π denotes its character. We have $\chi_\pi \in C(\Gamma)$, and the map $R(\Gamma) \ni \pi \mapsto \chi_\pi \in C(\Gamma)$ induces an isomorphism of \mathbf{C} -vector spaces $\mathcal{X} : R(\Gamma) \otimes_{\mathbf{R}} \mathbf{C} \cong C(\Gamma)$.

3.2

Consider a finite group Γ . Let $f : V \rightarrow W$ be an isomorphism of finite-dimensional Γ -Hilbert modules. If π is an irreducible representation of Γ , then let $V(\pi)$, $W(\pi)$ denote the π -isotypic components and $f(\pi) : V(\pi) \rightarrow W(\pi)$ the induced isomorphism. We define

$[[f]] \in R(\Gamma)$ by

$$[[f]](\pi) := \frac{1}{2 \dim(\pi)} \log |\det f(\pi)^* f(\pi)|.$$

Let

$$\mathcal{C} : \dots \rightarrow C^p \xrightarrow{c^p} C^{p+1} \rightarrow \dots$$

be an acyclic finite cochain complex of finite-dimensional Γ -Hilbert modules. Then there exists a chain contraction $\kappa^* : C^* \rightarrow C^{*-1}$, and $c^{ev} + \kappa^{ev} : C^{ev} \rightarrow C^{odd}$ is an isomorphism, where $C^{ev} := \bigoplus_{k \in \mathbf{Z}} C^{2k}$, $C^{odd} := \bigoplus_{k \in \mathbf{Z}} C^{2k+1}$. We define

$$\rho(\mathcal{C}) := \mathcal{X}[[c^{ev} + \kappa^{ev} : C^{ev} \rightarrow C^{odd}]].$$

Note that $\rho(\mathcal{C})$ does not depend on the choice of the chain contraction κ .

Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be a chain homotopy equivalence. Then we consider the complex $\text{cone}(f)$ with $\text{cone}(f)^n := C^n \oplus D^{n-1}$ and the differential

$$\begin{pmatrix} c^n & 0 \\ f^n & -d^{n-1} \end{pmatrix}.$$

We define $t(f) = \rho(\text{cone}(f)) \in C(\Gamma)$. Note that if $g : \mathcal{C} \rightarrow \mathcal{D}$ is homotopy equivalent to f , then $t(f) = t(g)$. If $g : \mathcal{D} \rightarrow \mathcal{E}$ is a second chain homotopy equivalence, then $t(g \circ f) = t(f) - t(g)$.

If \mathcal{C} is a finite complex of finite-dimensional Γ -Hilbert modules, then we consider the complex $\mathcal{H}(\mathcal{C})$ with p 'th space $H^p(\mathcal{C})$ and trivial differential. If $f : \mathcal{C} \rightarrow \mathcal{D}$ is a homotopy equivalence, then we obtain a homotopy equivalence $f_* : \mathcal{H}(\mathcal{C}) \rightarrow \mathcal{H}(\mathcal{D})$.

Let \mathcal{C}, \mathcal{D} be finite complexes of finite-dimensional Γ -Hilbert modules with preferred Γ -Hilbert module structures on $H^*(\mathcal{C}), H^*(\mathcal{D})$. A homotopy equivalence $f : \mathcal{C} \rightarrow \mathcal{D}$ is called simple if $t(f) + t(f_*) = 0$.

We call two finite complexes of finite-dimensional Γ -Hilbert modules \mathcal{C}, \mathcal{D} with preferred Γ -Hilbert module structures on $H^*(\mathcal{C}), H^*(\mathcal{D})$ equivalent, if there exists a simple homotopy equivalence $f : \mathcal{C} \rightarrow \mathcal{D}$. We write $[\mathcal{C}]$ for the equivalence class.

If \mathcal{C} is a finite complex of finite-dimensional Γ -Hilbert modules with preferred Γ -Hilbert module structure on $H^*(\mathcal{C})$, and if $i : \mathcal{H}(\mathcal{C}) \rightarrow \mathcal{C}$ is any embedding such that $i_* = \text{id}$, then we set $\rho(\mathcal{C}) := -t(i)$. If \mathcal{C} and \mathcal{D} are equivalent, then $\rho(\mathcal{C}) = \rho(\mathcal{D})$, hence we can write $\rho([\mathcal{C}]) := \rho(\mathcal{C})$, where \mathcal{C} is any representative of $[\mathcal{C}]$.

Let $\Gamma^0 \subset \Gamma$ be a subgroup such that Γ^0 acts trivially on $\mathcal{H}(\mathcal{C})$. Let $\hat{C}(\Gamma) := C(\Gamma)/C(\Gamma/\Gamma^0)$ and $\hat{\rho}(\mathcal{C})$ be the class of $\rho(\mathcal{C})$ in $\hat{C}(\Gamma)$.

Lemma 3.1 *$\hat{\rho}(\mathcal{C})$ is independent of the choice of the Γ -Hilbert module structure on $H^*(\mathcal{C})$.*

Proof. Let $\mathcal{H}_j(\mathcal{C})$, $j = 1, 2$, be the complex $\mathcal{H}(\mathcal{C})$ equipped with two Γ -Hilbert module structures. Let $\rho_j(\mathcal{C})$ be the corresponding torsion. Then we have $\rho_2(\mathcal{C}) = \rho_1(\mathcal{C}) - t(\text{id} : \mathcal{H}_2(\mathcal{C}) \rightarrow \mathcal{H}_1(\mathcal{C}))$. It is easy to see that $t(\text{id} : \mathcal{H}_2(\mathcal{C}) \rightarrow \mathcal{H}_1(\mathcal{C})) \in C(\Gamma/\Gamma^0)$. \square

3.3

Let Γ be a finite group and X be a Γ -space of the homotopy type of a finite Γ -CW-complex. Let \mathcal{F} be a Γ -equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We fix a Γ -Hilbert module structure on $H^*(X, \mathcal{F})$.

Let (Z, f) represent a simple structure on X . Then we form the cellular cochain complex $\mathcal{C}(Z, f^*\mathcal{F})$, which is a finite complex of finite-dimensional Γ -Hilbert modules. We equip $H^*(\mathcal{C}(Z, f^*\mathcal{F}))$ with the Γ -Hilbert module structure such that the canonical map $f^* : H^*(X, \mathcal{F}) \rightarrow H^*(\mathcal{C}(Z, f^*\mathcal{F}))$ becomes an isometry.

If (Z', f') and (Z, f) represent the same simple structure of X , then $\mathcal{C}(Z, f^*\mathcal{F})$ and $\mathcal{C}(Z', (f')^*\mathcal{F})$ are equivalent chain complexes. If X is a Γ -space with distinguished simple structure and Γ -Hilbert module structure on $H^*(X, \mathcal{F})$, then we write $[\mathcal{C}(X, \mathcal{F})] :=$

$[\mathcal{C}(Z, f^*\mathcal{F})]$, where $(Z, f^*\mathcal{F})$ is any representative of the distinguished simple structure of X . We define

$$\rho(X, \mathcal{F}) := \rho([\mathcal{C}(X, \mathcal{F})]) \in C(\Gamma) .$$

Let $\Gamma^0 \subset \Gamma$ act trivially on $H^*(X, \mathcal{F})$. Then the class $\hat{\rho}(X, \mathcal{F}) \in \hat{C}(\Gamma)$ does not depend on the choice of the Γ -Hilbert module structure on $H^*(X, \mathcal{F})$.

3.4

Let Γ be a finite group and $\Gamma' \subset \Gamma$. If Z is a Γ -CW complex, then since Γ is finite $\text{res}_{\Gamma'}^{\Gamma} Z$ carries a natural Γ' -CW structure. Moreover $\text{id} : \text{res}_{\Gamma'}^{\Gamma} Z \rightarrow \text{res}_{\Gamma'}^{\Gamma} Z$ represents the preferred simple structure given in Proposition 2.1 (1).

Let X be a Γ -space of the homotopy type of a finite Γ -CW complex. If (Z, f) represents a simple structure for X , then $(\text{res}_{\Gamma'}^{\Gamma} Z, f)$ represents a simple structure of $\text{res}_{\Gamma'}^{\Gamma} X$.

We choose a Γ -Hilbert module structure on $H^*(X, \mathcal{F})$ which we also use for $H^*(\text{res}_{\Gamma'}^{\Gamma} X, \mathcal{F}) = \text{res}_{\Gamma'}^{\Gamma} H^*(X, \mathcal{F})$.

Lemma 3.2

$$\text{res}_{\Gamma'}^{\Gamma} \rho(X, \mathcal{F}) = \rho(\text{res}_{\Gamma'}^{\Gamma} X, \mathcal{F}) .$$

Proof. We use $(\text{res}_{\Gamma'}^{\Gamma} Z, f)$ to represent the preferred simple structure of $\text{res}_{\Gamma'}^{\Gamma} X$. Then

$$\begin{aligned} \mathcal{C}(\text{res}_{\Gamma'}^{\Gamma} Z, f^*\mathcal{F}) &= \text{res}_{\Gamma'}^{\Gamma} \mathcal{C}(Z, f^*\mathcal{F}) \\ [\mathcal{C}(\text{res}_{\Gamma'}^{\Gamma} Z, f^*\mathcal{F})] &= [\text{res}_{\Gamma'}^{\Gamma} \mathcal{C}(Z, f^*\mathcal{F})] . \end{aligned} \tag{5}$$

Let $h : V \rightarrow W$ be an isomorphism of finite-dimensional Hilbert- Γ -modules and $\text{res}_{\Gamma'}^{\Gamma} h : \text{res}_{\Gamma'}^{\Gamma} V \rightarrow \text{res}_{\Gamma'}^{\Gamma} W$. Then for any irreducible representation τ of Γ' we have

$$[[\text{res}_{\Gamma'}^{\Gamma} h]](\tau) = \frac{1}{2 \dim(\tau)} \log |\det(\text{res}_{\Gamma'}^{\Gamma} h(\tau)^* \text{res}_{\Gamma'}^{\Gamma} h(\tau))|$$

$$\begin{aligned}
&= \sum_{\pi \in \hat{\Gamma}} \frac{[\pi : \tau]}{2 \dim(\pi)} \log |\det h^*(\pi) h(\pi)| \\
&= \sum_{\pi \in \hat{\Gamma}} [\pi : \tau] [[h]](\pi)
\end{aligned}$$

Note that $\text{res}_{\Gamma'}^{\Gamma} \chi_{\pi} = \sum_{\tau \in \hat{\Gamma}'} [\pi : \tau] \chi_{\tau}$. Thus

$$\begin{aligned}
\mathcal{X}([[\text{res}_{\Gamma'}^{\Gamma} h]]) &= \sum_{\tau \in \hat{\Gamma}'} [[\text{res}_{\Gamma'}^{\Gamma} h]](\tau) \chi_{\tau} \\
&= \sum_{\tau \in \hat{\Gamma}'} \sum_{\pi \in \hat{\Gamma}} [\pi : \tau] [[h]](\pi) \chi_{\tau} \\
&= \sum_{\pi \in \hat{\Gamma}} [[h]](\pi) \text{res}_{\Gamma'}^{\Gamma} \chi_{\pi} \\
&= \text{res}_{\Gamma'}^{\Gamma} \mathcal{X}([h]).
\end{aligned} \tag{6}$$

The Lemma now follows from (5) and (6). \square

3.5

Let G be a compact Lie group, X be a finite G -CW-complex and \mathcal{F} be a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We choose a G -Hilbert module structure on $H^*(X, \mathcal{F})$ which induces a Γ -Hilbert module structure on $H^*(\text{res}_{\Gamma}^G X, \mathcal{F}) = \text{res}_{\Gamma}^G H^*(X, \mathcal{F})$ for all $G/\Gamma \in \text{Or}_f(G)$.

If $G/\Gamma \in \text{Or}_f(G)$, then $\text{res}_{\Gamma}^G X$ has a preferred simple structure by Proposition 2.1 (1) and $\rho(\text{res}_{\Gamma}^G X, \mathcal{F}) \in C(\Gamma)$ is defined. If $h \in G$, then let $\Gamma^h := h\Gamma h^{-1}$, $G/\Gamma^h \in \text{Or}_f(G)$, and set $g^h := hgh^{-1}$ for $g \in G$.

Lemma 3.3 *If $g \in \Gamma$ and $h \in G$, then $\rho(\text{res}_{\Gamma}^G X, \mathcal{F})(g) = \rho(\text{res}_{\Gamma^h}^G X, \mathcal{F})(g^h)$.*

Proof. Let (Z, f) represent the preferred simple structure of $\text{res}_{\Gamma}^G X$. Then by Proposition 2.1 (4) (we employ the notation introduced there) the pair (Z^h, f^h) represents the preferred

simple structure of $\text{res}_{\Gamma^h}^G X$. We have an isomorphism of complexes of Γ^h -Hilbert modules $C(Z, f^* \mathcal{F})^h = C(Z^h, (f^h)^* \mathcal{F})$, where the underlying space of $C(Z, f^* \mathcal{F})^h$ is $C(Z, f^* \mathcal{F})$ and Γ^h acts by $g^h \mapsto g$. Similarly we have isomorphisms of complexes Γ^h -Hilbert modules $\mathcal{H}(\text{res}_{\Gamma}^G X, \mathcal{F})^h = \mathcal{H}(\text{res}_{\Gamma^h}^G X, \mathcal{F})$. If i, i^h, j denote the inclusions $i : \mathcal{H}(\text{res}_{\Gamma}^G X, \mathcal{F}) \hookrightarrow C(Z, f^* \mathcal{F})$, $i^h : \mathcal{H}(\text{res}_{\Gamma}^G X, \mathcal{F})^h \hookrightarrow C(Z, f^* \mathcal{F})^h$, $j : \mathcal{H}(\text{res}_{\Gamma^h}^G X, \mathcal{F}) \hookrightarrow C(Z^h, (f^h)^* \mathcal{F})$, then we have $t(i)(g) = t(i^h)(g^h) = t(j)(g^h)$. This proves the Lemma. \square

3.6

We keep the assumptions of 3.5 Then the Lemmas 3.3 and 3.2 imply

Corollary 3.4 *The collection $\{\rho(\text{res}_{\Gamma}^G X, \mathcal{F})\}_{G/\Gamma \in \text{Or}_f(G)}$, defines a section of the functor C (see Subsection 1.4).*

Let now M be a closed G -manifold and $F \rightarrow M$ be a G -equivariant flat hermitean vector bundle. We fix G -Hilbert module structures on $H^*(M, \mathcal{F})$. If $G/\Gamma \in \text{Or}_f(G)$, then $\text{res}_{\Gamma}^G M$ has a preferred simple structure. We employ this structure in order to define $\rho(\text{res}_{\Gamma}^G M, \mathcal{F})$ as explained in Section 3.3.

Proposition 3.5 *The collection $\{\rho(\text{res}_{\Gamma}^G M, \mathcal{F})\}_{G/\Gamma \in \text{Or}_f(G)}$, defines a section of the functor C .*

Proof. Let $f : X \rightarrow M$ represent the preferred simple structure. Then by Proposition 2.1, (3), we have $\rho(\text{res}_{\Gamma}^G M, \mathcal{F}) = \rho(\text{res}_{\Gamma}^G X, f^* \mathcal{F})$. Thus the Proposition is implied by Corollary 3.4. \square

4 Computations

4.1

Let G be a compact Lie group and X be a finite G -CW-complex. Let \mathcal{F} be a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces. We choose a G -Hilbert module structure on $H^*(X, \mathcal{F})$. Being a G -CW complex X has a natural filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \dots \subset X_N = X$. Consider $G/\Gamma \in \text{Or}_f(G)$.

Lemma 4.1 *There exists a representative $f : Z \rightarrow \text{res}_\Gamma^G X$ of the preferred simple structure such that*

(1) : *Z is filtered by Γ -CW subcomplexes $\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset \dots \subset Z_N = Z$ and $f|_{Z_p} : Z_p \rightarrow \text{res}_\Gamma^G X_p$ represents the preferred simple structure for all $p \in \{0, 1, \dots, N\}$.*

(2) *If*

$$\dots \subset F_{p+1}\mathcal{C}(Z, f^*\mathcal{F}) \subset F_p\mathcal{C}(Z, f^*\mathcal{F}) \subset \dots$$

denotes the decreasing filtration of the associated cochain complexes (we write $F_p := F_p\mathcal{C}(Z, f^\mathcal{F})$), then F_p/F_{p+1} is the cochain complex associated to a representative of the preferred simple structure of $\sqcup_{i \in I_p} \text{res}_\Gamma^G G/H_i \times (D^p, S^{p-1})$ and the local system $Q^*\mathcal{F}$, where Q is given by*

$$Q := \sqcup_{i \in I_p} Q_i : \sqcup_{i \in I_p} G/H_i \times D^p \rightarrow X_p$$

(recall that I_p is the indexing set of the p -dimensional G -cells of X and Q_i denote characteristic maps).

Proof. The construction of $f : Z \rightarrow \text{res}_\Gamma^G X$ goes by induction and is based on [8], 4.29-4.32. For X_{-1} the assertion is trivial. Assume that we have constructed the simple structure (Z_{n-1}, f_{n-1}) of $\text{res}_\Gamma^G X_{n-1}$ together with the filtration by Γ -subspaces $(Z_{n-1})_m$. Then we have to construct a simple structure (Z_n, f_n) of $\text{res}_\Gamma^G X_n$ together with the filtration by Γ -subspaces $(Z_n)_m$.

Let X_n be given by the G -push out

$$\begin{array}{ccc} \sqcup_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\sqcup_i q_i} & X_{n-1} \\ \downarrow & & \downarrow \\ \sqcup_{i \in I_n} G/H_i \times D^n & \xrightarrow{\sqcup_i Q_i} & X_n \end{array}.$$

We choose representatives $h_i : V_i \rightarrow \text{res}_\Gamma^G G/H_i$ of the preferred simple structures. Then $h_i \times \text{id} : U_i := V_i \times D^n \rightarrow \text{res}_\Gamma^G G/H_i \times D^n$ is a simple structure on the cell $\text{res}_\Gamma^G G/H_i \times D^n$. We choose a Γ -equivariant cellular map $p : \sqcup_{i \in I_n} U_i \times S^{n-1} \rightarrow Z_{n-1}$ such that $f_{n-1} \circ p \sim_\Gamma \sqcup_{i \in I_n} q_i \circ l_i$ and \sim_Γ stands for Γ -homotopic. We now replace (Z_{n-1}, f_{n-1}) by $(\text{Cyl}(p), f'_{n-1})$, which represents the same simple structure on X_{n-1} (f'_{n-1} has still to be constructed). The filtration of $\text{Cyl}(p) = \sqcup_{i \in I_n} U_i \times S^{n-1} \times I \cup_p Z_{n-1}$ is given by $\text{Cyl}(p)_m = (Z_{n-1})_m \subset \text{Cyl}(p)$ for $m \leq n-2$. Let $\text{pr} : \text{Cyl}(p) \rightarrow Z_{n-1}$ be the projection. In order to construct f'_{n-1} we consider the following Γ -homotopy commutative diagram

$$\begin{array}{ccc} \sqcup_{i \in I_n} \text{res}_\Gamma^G G/H_i \times S^{n-1} \sqcup X_{n-2} & \xrightarrow{\sqcup_i q_i \sqcup j} & \text{res}_\Gamma^G X_{n-1} \\ \uparrow \sqcup_i l_i \sqcup f_{n-2} & & \uparrow f_{n-1} \circ \text{pr} \\ \sqcup_{i \in I_n} U_i \times S^{n-1} \sqcup \text{Cyl}(p)_{n-2} & \xrightarrow{J} & \text{Cyl}(p) \end{array},$$

where $j : X_{n-2} \hookrightarrow X_{n-1}$ is the inclusion, $l_i := (h \times \text{id})|_{U_i \times S^{n-1}}$, and $J|_{\sqcup_{i \in I_n} U_i \times S^{n-1}}$ is the natural identification with the closed subspace $\sqcup_{i \in I_n} U_i \times S^{n-1} \times \{0\} \subset \sqcup_{i \in I_n} U_i \times S^{n-1} \times I$. Since J is a cofibration we can find $f'_{n-1} \sim_\Gamma f_{n-1} \circ \text{pr}$ such that $f_{n-2} \circ J|_{\text{Cyl}(p)_{n-2}} = (f'_{n-1})|_{\text{Cyl}(p)_{n-2}}$ and the following diagram commutes:

$$\begin{array}{ccccc} \sqcup_{i \in I_n} \text{res}_\Gamma^G G/H_i \times D^n & \hookleftarrow & \sqcup_{i \in I_n} \text{res}_\Gamma^G G/H_i \times S^{n-1} & \xrightarrow{\sqcup_i q_i} & \text{res}_\Gamma^G X_{n-1} \\ \uparrow & & \uparrow & & \uparrow f'_{n-1} \\ \sqcup_{i \in I_n} U_i \times D^n & \hookleftarrow & \sqcup_{i \in I_n} U_i \times S^{n-1} & \xrightarrow{J} & \text{Cyl}(p) \end{array}.$$

Let Z_n be the Γ -push out

$$\begin{array}{ccc} \sqcup_{i \in I_n} U_i \times S^{n-1} & \rightarrow & \text{Cyl}(p) \\ \downarrow & & \downarrow \\ \sqcup_{i \in I_n} U_i \times D^n & \rightarrow & Z_n \end{array}$$

and $f_n : Z_n \rightarrow \text{res}_\Gamma^G X_n$ be the natural map of push outs. Then $f_n : Z_n \rightarrow \text{res}_\Gamma^G X_n$ represents the preferred simple structure, and Z_n is filtered by Γ -subspaces $(Z_n)_m$ such that $(f_n)|_{(Z_n)_m} : (Z_n)_m \rightarrow \text{res}_\Gamma^G X_m$ represents the preferred simple structure for all $m \leq n$. This finishes the proof of **(1)**. Assertion **(2)** is an easy consequence of the construction. \square

4.2

For $i \in I_p$ define

$$\bar{Q}_i : G/H_i \cong G/H_i \times \{0\} \hookrightarrow G/H_i \times D^p \xrightarrow{Q_i} X$$

and set $\mathcal{F}_i := \bar{Q}_i^* \mathcal{F}$. For any $p \in \mathbf{N}_0$ and $i \in I_p$ we fix G -Hilbert module structures on $H^*(G/H_i, \mathcal{F}_i)$. This induces G -Hilbert module structures on the cohomology complexes $\mathcal{H}(G/H_i, \mathcal{F}_i)$. If \mathcal{C} is a cochain complex, then let $\mathcal{C}[p]$ be the cochain complex with $\mathcal{C}[p]^n := \mathcal{C}^{n+p}$ obtained from \mathcal{C} . We equip $H^*(F_p/F_{p+1})$ with Γ -Hilbert module structures such that the natural isomorphism $\mathcal{H}(F_p/F_{p+1}) \cong \oplus_{i \in I_p} \mathcal{H}(G/H_i, \mathcal{F}_i)[p]$ becomes an isometry. Then we have

$$[F_p/F_{p-1}] = \oplus_{i \in I_p} [\mathcal{C}(U_i, h_i^* \mathcal{F}_i)[p]] .$$

We are going to express $\rho(\text{res}_\Gamma^G X, \mathcal{F})$ in terms of $\rho(\text{res}_\Gamma^G G/H_i, \mathcal{F}_i)$ and a contribution of the spectral sequence $\mathcal{E} := (E_r^{p,q}, d_r)$ associated to the filtration of X . For trivial Γ this was worked out in [9]. But [9], Lemmas 4.6 and 4.7, extend immediately to the case of a finite group Γ . We recall the result. Let

$$\begin{aligned} Z_r^{p,q} &:= \text{im}(H^{p+q}(F_p/F_{p+r}) \rightarrow H^{p+q}(F_p/F_{p+1})) \\ B_r^{p,q} &:= \text{im}(H^{p+q-1}(F_{p-r+1}/F_p) \rightarrow H^{p+q}(F_p/F_{p+1})) \\ E_r^{p,q} &:= Z_r^{p,q}/B_r^{p,q} \\ Z_\infty^{p,q} &:= \text{im}(H^{p+q}(F_p) \rightarrow H^{p+q}(F_p/F_{p+1})) \\ B_\infty^{p,q} &:= \text{im}(H^{p+q-1}(\mathcal{C}(Z, f^* \mathcal{F})) \rightarrow H^{p+q}(F_p/F_{p+1})) \\ E_\infty^{p,q} &:= Z_\infty^{p,q}/B_\infty^{p,q} \\ F^{p,q} &:= \text{im}(H^{p+q}(F_p) \rightarrow H^{p+q}(\mathcal{C}(Z, f^* \mathcal{F}))) . \end{aligned}$$

There are natural isomorphisms $\psi^{p,q} : F^{p,q}/F^{p+1,q-1} \rightarrow E_\infty^{p,q}$.

Note that $H^*(\mathcal{C}(Z, f^* \mathcal{F})) \xrightarrow{f^*} H^*(X, \mathcal{F})$ has a preferred Γ -Hilbert module structure. We equip $Z_r^{p,q}, B_r^{p,q}, E_r^{p,q}, Z_\infty^{p,q}, B_\infty^{p,q}, E_\infty^{p,q}, F^{p,q}$ with the corresponding (sub)quotient Γ -Hilbert module structures. For any p, q, r we have a complex of finite Γ -Hilbert modules

$$\mathcal{E}_r^{p,q} : \dots \rightarrow E_r^{p+nr, q-(r-1)n} \rightarrow E_r^{p+(n+1)r, q-(r-1)(n+1)} \rightarrow \dots ,$$

and $H^n(\mathcal{E}_r^{p,q}) = \mathcal{E}_{r+1}^{p+nr, q-(r-1)n}$ has a preferred Γ -Hilbert module structure. Thus $\rho(\mathcal{E}_r^{p,q})$ is well defined. Furthermore note that $\rho(F_p/F_{p+1}) = (-1)^p \sum_{i \in I_p} \rho(\text{res}_\Gamma^G G/H_i, \mathcal{F}_i)$. The following Proposition can be proved by repeating the argument of the proof of [9], Thm. 4.4.

Proposition 4.2

$$\rho(\text{res}_\Gamma^G X, \mathcal{F}) = \sum_p (-1)^p \sum_{i \in I_p} \rho(\text{res}_\Gamma^G G/H_i, \mathcal{F}_i) + \sum_{r \geq 1} \sum_{p=0}^{r-1} \sum_q (-1)^{p+q} \rho(\mathcal{E}_r^{p,q}) - \sum_{p,q} (-1)^{p+q} \mathcal{X}[[\psi^{p,q}]] .$$

Note that all three terms of the right-hand side may depend on the choice of the preferred Γ -Hilbert module structures on $H^*(G/H_i, \mathcal{F}_i)$, while the left hand side does not.

4.3

Let X be a finite G -CW-complex and \mathcal{F} a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces. Let $G/\Gamma \in \text{Or}_f(G)$. Then $\hat{\rho}(\text{res}_\Gamma^G X, \mathcal{F}) \in \hat{C}(\Gamma)$ is well defined.

Proposition 4.3

$$\hat{\rho}(\text{res}_\Gamma^G X, \mathcal{F}) = \sum_p (-1)^p \sum_{i \in I_p} \hat{\rho}(\text{res}_\Gamma^G G/H_i, \mathcal{F}_i)$$

Proof. We fix G -Hilbert module structures on $H^*(X, \mathcal{F})$ and $H^*(G/H_i, \mathcal{F}_i)$. Let $q : \Gamma \rightarrow \Gamma/\Gamma^0$ be the projection. We have to show that

$$\rho(\mathcal{E}_r^{p,q}) \in q^*C(\Gamma/\Gamma^0), \quad \mathcal{X}[[\psi^{p,q}]] \in q^*C(\Gamma/\Gamma^0) . \quad (7)$$

Viewing X as a filtered G -space we see that there is a spectral sequence $\tilde{\mathcal{E}} := (\tilde{E}_r^{p,q}, d_r)$ of G -modules with

$$\tilde{Z}_r^{p,q} := \text{im}(H^{p+q}(X_{p+r-1}, X_{p-1}) \rightarrow \oplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) ,$$

$$\begin{aligned}
\tilde{B}_r^{p,q} &:= \text{im}(H^{p+q-1}(X_{p-1}/X_{p-r}) \rightarrow \oplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) , \\
\tilde{E}_r^{p,q} &:= \tilde{Z}_r^{p,q} / \tilde{B}_r^{p,q} \\
\tilde{Z}_\infty^{p,q} &:= \text{im}(H^{p+q}(X, X_{p-1}, \mathcal{F}) \rightarrow \oplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) \\
\tilde{B}_\infty^{p,q} &:= \text{im}(H^{p+q-1}(X, \mathcal{F}) \rightarrow \oplus_{i \in I_p} H^{p+q}(G/H_i, \mathcal{F}_i)) \\
\tilde{E}_\infty^{p,q} &:= \tilde{Z}_\infty^{p,q} / \tilde{B}_\infty^{p,q} \\
\tilde{F}^{p,q} &:= \text{im}(H^{p+q}(X, X_{p-1}, \mathcal{F}) \rightarrow H^{p+q}(X, \mathcal{F}))
\end{aligned}$$

and G -equivariant maps $\tilde{\psi}^{p,q} : \tilde{F}^{p,q} / \tilde{F}^{p+1,q-1} \rightarrow \tilde{E}_\infty^{p,q}$ such that $\mathcal{E} = \text{res}_\Gamma^G \tilde{\mathcal{E}}$, $\psi^{p,q} = \text{res}_\Gamma^G \tilde{\psi}^{p,q}$. Consider the exact sequence

$$0 \rightarrow G^0 \rightarrow G \xrightarrow{q} \pi_0(G) \rightarrow 0 .$$

Observe that the representation of G on $\tilde{\mathcal{E}}$ and $\tilde{F}^{p,q}$ factors over q . In particular Γ^0 is represented trivially. Thus (7) follows. \square

Fix G -Hilbert module structures on $H^*(X, \mathcal{F})$ and $H^*(G/H_i, \mathcal{F}_i)$. Then Proposition 4.3 has the

Corollary 4.4

$$\hat{\rho}(X, \mathcal{F}) = \sum_p (-1)^p \sum_{i \in I_p} \hat{\rho}(G/H_i, \mathcal{F}_i) .$$

5 Reduction to Cartan subgroups

5.1

Let G be a compact Lie group, X be a finite G -CW complex, and let $F \rightarrow X$ be an equivariant flat hermitean vector bundle.

Recall that a Cartan subgroup T of G is a topologically cyclic closed subgroup such that the Weyl group $N_G(T)/T$ is finite, where $N_G(T)$ is the normalizer of T in G . If $T \subset G$ is a Cartan subgroup then it is isomorphic to the product of a torus and a finite cyclic group (see [2], p.177 ff, for more details about Cartan subgroups).

If $g \in G$, then there exists a Cartan subgroup containing g . The conjugacy classes of Cartan subgroups are in natural bijection with the cyclic subgroups of $\pi_0(G)$.

Let $\{T_i\}_{i \in C}$ be a set of representatives of conjugacy classes of Cartan subgroups of G . Then restriction defines inclusions

$$\oplus_{i \in C} \text{res}_{T_i}^G : C(FG) \rightarrow \oplus_{i \in C} C(FT_i), \quad \oplus_{i \in C} \hat{\text{res}}_{T_i}^G : \hat{C}(FG) \rightarrow \oplus_{i \in C} \hat{C}(FT_i).$$

In order to determine $f \in \hat{C}(FG)$ it is thus sufficient to compute $\hat{\text{res}}_{T_i}^G f \in \hat{C}(FT_i)$ for all $i \in C$.

Let T be any Cartan subgroup of G and let $f : Z \rightarrow \text{res}_T^G X$ represent the preferred simple structure. Then $\hat{\text{res}}_T^G \hat{\rho}(X, \mathcal{F}) = \hat{\rho}(Z, f^* \mathcal{F})$.

5.2

We further study the contribution of the T -cells of Z . Let $S \subset T$ be any closed subgroup, and let \mathcal{F} be a T -equivariant local system on T/S . Then $\rho(T/S, \mathcal{F}) \in \hat{C}(FT)$ is well defined.

Lemma 5.1 *If $\dim(T/S) \neq 1$, then $\hat{\rho}(T/S, \mathcal{F}) \in \hat{C}(FT) = 0$.*

Proof. Assume first that T/S is even-dimensional. Then T/S is orientable. Let $t \in FT$ generate the finite group $H \subset \text{Aut}(T/S)$. Then H acts by orientation-preserving diffeomorphisms on T/S . We employ [10], Prop. 3.23, which says that $\hat{\rho}(\text{res}_H^T T/S, \mathcal{F})$ can be derived from the Poincaré torsion $\rho_{pd}^H(T/S, \mathcal{F})$ of the H -manifold $\text{res}_H^T T/S$ (see loc.cit.

Definition 3.19). Since T is abelian, we see that H acts freely or trivially on T/S . In both cases $\rho_{pd}^H(T/S, \mathcal{F})$ vanishes (see loc.cit. Prop. 3.20), and thus $\text{r\^es}_H^T \hat{\rho}(T/S, \mathcal{F}) = 0$. Since t was arbitrary we conclude that $\hat{\rho}(T/S, \mathcal{F}) = 0$.

Now we consider the case that T/S is odd-dimensional. We fix a T -invariant Riemannian metric $g^{T/S}$ on T/S and equip $H^*(T/S, \mathcal{F})$ with the T -module structure such that the de Rham isomorphism becomes an isometry. Then $\rho(T/S, \mathcal{F}) \in C(FT)$ is well-defined.

By Lemma 1.2 we have $\rho(T/S, \mathcal{F}) = \rho_{an}(T/S, g^{T/S}, \mathcal{F})|_{FT}$. If $\dim(T/S) > 1$, then we find two everywhere linearly independent unit-length Killing vector fields in $C^\infty(T/S, T(T/S))$. Using the induced decomposition of the de Rham complex by a standard argument $\rho_{an}(T/S, g^{T/S}, \mathcal{F}) = 0$. This implies the lemma. \square

5.3

Let $S \subset T$ be any closed subgroup such that $\dim(T/S) = 1$, and let \mathcal{F} be a T -equivariant locally constant sheaf of Hilbert spaces. Since T is abelian and $\rho(T/S, \mathcal{F})$ is additive with respect to \mathcal{F} without loss of generality we can assume that \mathcal{F} is one-dimensional.

We fix an orientation on T/S . The space T/S is a disjoint union of oriented circles C_1, \dots, C_s . Let $e^{2\pi i a_i} \in S^1$ denote the holonomy of \mathcal{F} on C_i . We choose the Riemannian metric $g^{T/S}$ such that the circles have length 1.

Let $t \in T$. We distinguish two cases

- a) that $tC_i = C_i$ for all i
- b) that $tC_i \neq C_i$ for all i .

In case b) we have $\rho_{an}(T/S, g^{T/S}, \mathcal{F})(t) = 0$. In case a) we have $\rho_{an}(T/S, g^{T/S}, \mathcal{F})(t) = \sum_{i=1}^s \rho_{an}(C_i, g^{C_i}, \mathcal{F}_i)(t)$, where \mathcal{F}_i is the restriction of \mathcal{F} to C_i .

Thus we are reduced to the following situation. Let C_i be a circle, $F \rightarrow C_i$ be a one-dimensional flat hermitean bundle over C_i , and t be an automorphism of $F \rightarrow C_i$, which acts as rotation on C_i . We parametrize $C_i = \mathbf{R}/\mathbf{Z}$ with coordinate x . Then $t(x) = x + \tau_i$, $\tau_i \in \mathbf{R}/\mathbf{Z}$. Sections of $F|_{C_i}$ are identified with functions $f : \mathbf{R} \rightarrow \mathbf{C}$ satisfying $f(x+1) = e^{2\pi i a_i} f(x)$. Then $(tf)(x) = \lambda_i f(x - \tau_i)$ for certain $\lambda_i \in S^1$.

We have $\rho(C_i, g^{C_i}, \mathcal{F})_{an}(t) = \psi(\lambda_i, a_i, \tau_i)$, and ψ will be determined in Subsection 5.4. We obtain $\rho_{an}(T/S, g^{T/S}, \mathcal{F})(t) = \sum_{i=1}^s \psi(\lambda_i, a_i, \tau_i)$. If \mathcal{F} is higher-dimensional, then λ_i and a_i become commuting diagonalizable matrices, and we have $\rho_{an}(T/S, g^{T/S}, \mathcal{F})(t) = \sum_{i=1}^s \text{Tr } \psi(\lambda_i, a_i, \tau_i)$.

5.4

In this Subsection we derive a formula for $\psi(\lambda, a, \tau)$. Let C be a circle, $F \rightarrow C$ be a one-dimensional flat hermitean bundle over C , and t be an automorphism of $F \rightarrow C$, which acts as rotation on C .

We parametrize $C = \mathbf{R}/\mathbf{Z}$ with coordinate x . Then $t(x) = x + \tau$, $\tau \in \mathbf{R}/\mathbf{Z}$. Sections of F are identified with functions $f : \mathbf{R} \rightarrow \mathbf{C}$ satisfying $f(x+1) = e^{2\pi i a} f(x)$. Then $(tf)(x) = \lambda f(x - \tau)$ for certain $\lambda \in S^1$.

We identify one-forms on C with functions using the basis $dx \in C^\infty(C, T^*C)$. Let $\Delta_F = -(d/dx)^2$ be the Laplace operator. The eigenvectors of Δ_F are $f_n(x) = \exp(2\pi i(n+a)x)$, and the corresponding eigenvalue is $\mu_n = 4\pi^2(n+a)^2$. The action of t on the eigenspace spanned by f_n is multiplication by $\lambda e^{-2\pi i(n+a)\tau}$.

First assume that $a \in (0, 1)$. Let

$$F_n(s) := \frac{\lambda e^{-2\pi i(n+a)\tau}}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-4\pi^2(n+a)^2 t} dt = \frac{\lambda e^{-2\pi i(n+a)\tau}}{4^s \pi^{2s} (n+a)^{2s}},$$

then

$$\psi(\lambda, a, \tau) = -\frac{d}{ds} \Big|_{s=0} \sum_{n \in \mathbf{Z}} F_n(s),$$

where we employ a meromorphic continuation of the sum $\sum_{n \in \mathbf{Z}} F_n(s)$ which converges for $\operatorname{Re}(s) > 1$.

Let $\phi(y, a, s) := \sum_{n=0}^{\infty} \frac{e^{2\pi i y n}}{(n+a)^s}$. Then

$$\psi(\lambda, a, \tau) = -\frac{d}{ds} \Big|_{s=0} \frac{\lambda}{4^s \pi^{2s}} \left(e^{-2\pi i \tau a} \phi(-\tau, a, 2s) + e^{-2\pi i \tau(a-1)} \phi(\tau, 1-a, 2s) \right) .$$

If $a = 0$ and $\tau \in (0, 1)$, then with $\phi(y, s) := \sum_{n=1}^{\infty} \frac{e^{2\pi i y n}}{n^s}$ we obtain

$$\psi(\lambda, 0, \tau) = -\frac{d}{ds} \Big|_{s=0} \frac{\lambda}{4^s \pi^{2s}} (\phi(\tau, 2s) + \phi(-\tau, 2s)) .$$

Using [7], Prop. 31, one can show that

$$\psi(\lambda, 0, \tau) = \lambda \mathbf{D} + \lambda \left(\frac{\Gamma(\tau)'}{\Gamma(\tau)} + \frac{\Gamma(1-\tau)'}{\Gamma(1-\tau)} \right)$$

for some explicitly known constant $\mathbf{D} \in \mathbf{C}$. In the case that $a = 0$ and $\tau = 0$ we have

$$\psi(\lambda, 0, 0) = -\frac{d}{ds} \Big|_{s=0} \frac{2\lambda \zeta_R(2s)}{4^s \pi^{2s}} ,$$

where ζ_R denotes the Riemann zeta function.

5.5

In this Subsection we combine the results of Subsections 5.1, 5.2, and 5.3. For simplicity we assume that G is a connected compact Lie group. Let X be a finite G -CW complex, and let $\mathcal{F} \rightarrow X$ be a G -equivariant locally constant sheaf of finite-dimensional Hilbert spaces. Let $T \subset G$ be a maximal torus, and let $f : Z \rightarrow \operatorname{res}_T^G X$ represent the preferred simple structure.

Let I be the index set for the T -cells $E = T/S_E \times D^{n_E}$ of Z with $\dim(T/S_E) = 1$. Since T is connected the quotient T/S is a circle S^1 . If $t \in T$, then for each $E \in I$ we fix an orientation of T/S_E , and we define the rotation number $\tau_E(t) \in \mathbf{R}/\mathbf{Z}$, the constant $\lambda_E(t)$, and the holonomy $e^{2\pi i a_E}$ of \mathcal{F}_E . We obtain the following Proposition.

Proposition 5.2 $\hat{\rho}(X, \mathcal{F})$ is uniquely determined by its restriction to T , which is represented by the function

$$FT \ni t \mapsto \sum_{E \in I} (-1)^{n_E} \text{Tr } \psi(\lambda_E(t), a_E, \tau_E(t)) .$$

5.6

In this section we compute the equivariant Reidemeister torsion of odd-dimensional symmetric spaces. We recover results of [6] using a completely different method. Let θ be the constant sheaf with fibre \mathbf{C} , and let $\hat{\rho}(M) := \hat{\rho}(M, \theta)$ for any closed G -manifold M , where we equip θ with the obvious G -action.

Let G/K be a compact, irreducible, odd-dimensional symmetric space. We assume that G is connected.

Lemma 5.3 *If $\hat{\rho}(G/K) \neq 0$, then $\text{rank} G = \text{rank} K + 1$, and*

$$G/K = \begin{cases} SO(2m)/SO(2p-1) \times SO(2m-2p+1) & \text{or} \\ SU(3)/SO(3) \end{cases} .$$

Proof. Let T be a maximal torus of G and $f : X \rightarrow \text{res}_T^G(G/K)$ be a representative of the preferred simple structure. Using the construction given in [8], 4.36, we can choose X such that it has the same set of T -orbit types as G/K . If $\hat{\rho}(G/K) = \hat{\rho}(X) \neq 0$, then by Proposition 5.2 there exists a one-dimensional T -orbit $TgK \subset G/K$. Then $T^{g^{-1}} \cap K$ is a $\text{rank} G - 1$ -dimensional torus in K . Hence $\text{rank} G \geq \text{rank} K \geq \text{rank} G - 1$. If $\text{rank} K = \text{rank} G$, then G/K is even-dimensional. Thus $\text{rank} K = \text{rank} G - 1$. The second assertion follows from the classification of irreducible compact symmetric spaces. \square

Now assume that G/K is a compact, irreducible, odd-dimensional symmetric space with $\text{rank} K = \text{rank} G - 1$.

Lemma 5.4 *There is a one-to-one correspondence of one-dimensional T -orbits in G/K and $W_G(T)/W_K(T)$ given by $N_G(T) \ni g \mapsto Tg^{-1}K$, where $W_G(T) := N_G(T)/T$ and $W_K(T) := N_K(T)/T \cap K$.*

Proof. We can assume that $T \cap K = S$ is a maximal torus of K . If $T' \subset G$ is a second maximal torus of G with $T' \cap K = S$, then $T' = T$. Indeed, on the level of Lie algebras we have $t' = s \oplus (k^\perp)^S = t$.

Let TgK be a one-dimensional T -orbit in G/K . Then $T^{g^{-1}} \cap K$ is a maximal torus in K , and hence $T^{g^{-1}} \cap K = S^k$ for a suitable $k \in K$. Replacing g by gk^{-1} we obtain $T^{g^{-1}} \cap K = S$, and thus $g^{-1} \in N_G(T)$. If $g \in T \cup K$, then $TgK = TK$. Thus the correspondence of $W_G(T)/W_K(T)$ with the set of one-dimensional orbits is established by $g^{-1} \in N_G(T) \mapsto TgK$. \square

Let $s \subset t$ be the Lie algebra of $S \subset T$. Let $a := t/s$, and let $L \subset a$ be the lattice of those $[l] \in t/s$, $l \in t$, which satisfy $\exp(l) \in S$. We identify $a \cong \mathbf{R}$ such that L is identified with \mathbf{Z} . The exponential map yields an identification $i : \mathbf{R}/\mathbf{Z} \cong a/L \cong T/S$. If $t \in T$, then let $\alpha(t) \in \mathbf{R}/\mathbf{Z}$ be such that $ti(x) = i(\alpha(t) + x)$, $\forall x \in \mathbf{R}$.

For $\tau \in \mathbf{R}/\mathbf{Z}$, $\tau \neq 0$, define

$$\psi(\tau) := \mathbf{D} + \left(\frac{\Gamma(\hat{\tau})'}{\Gamma(\hat{\tau})} + \frac{\Gamma(1 - \hat{\tau})'}{\Gamma(1 - \hat{\tau})} \right),$$

where $\hat{\tau} \in (0, 1)$ represents τ . If $\tau = 0$, we put

$$\psi(\tau) := -\frac{d}{ds} \Big|_{s=0} \frac{2\lambda\zeta_R(2s)}{4^s\pi^{2s}}.$$

Because of the equality (4) the following proposition recovers the computation of equivariant analytic torsion of compact symmetric spaces [6], Thm. 11, up to a constant function.

Proposition 5.5 *Let G/K be a compact, irreducible, odd-dimensional symmetric space. If $\text{rank} K \neq \text{rank} G - 1$, then $\hat{\rho}(G/K) = 0$. If $\text{rank} K = \text{rank} G - 1$, then let $T \subset G$ be a maximal torus such that $S := T \cap K$ is a maximal torus of K . Then the restriction of $\hat{\rho}(G/K)$ to T is represented by*

$$FT \ni t \mapsto \frac{1}{\#W_K(T)} \sum_{w \in W_G(T)} \psi(\alpha(t^w)) .$$

Proof. Let $f : X \rightarrow \text{res}_{N_G(T)}^G(G/K)$ be a representative of the natural simple structure. By Lemma 5.4 there is exactly one isolated one-dimensional $N_G(T)$ -orbit in G/K . Constructing X by the inductive procedure given in [8], 4.36, we can assume that X has exactly one cell $E = N_G(T)/N_K(T) \times D^{n_E}$ with one-dimensional $N_G(T)$ -orbits. Moreover $n_E = 0$.

The T -space $\text{res}_T^{N_G(T)}(X)$ has a natural T -CW structure (since $N_G(T)/T$ is finite), and $\text{res}_T^{N_G(T)}(E)$ is the only T -cell with one-dimensional T -orbits. But $\text{res}_T^{N_G(T)}(E)$ is the disjoint union of spaces T/S^g , $g \in W_G(T)/W_K(T)$. The proposition now follows from Proposition 5.2. \square

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